Homework 9

Projections

Quite often, it is stated as a definition that a matrix P is the matrix of a projection iff

$$P^2 = P \qquad (\textit{idempotent})$$

$$P^T = P \qquad (\textit{symmetric})$$

but that is indeed the definition of an orthogonal projection.

There is a more general one, where the projection is done along a subspace K onto another subspace S.

These two subspaces are not required to be orthogonal (that is any vector in K be orthogonal to any vector in S), but only to be complements, meaning their intersection contains only the nul vector,

$$S \cap K = \{\mathbf{0}\}$$

and the whole space is a direct sum of K and S. There is one and only one way to write any vector v as a sum of two vectors, one belonging to S, and the other one to K:

$$v = a + k, \qquad \text{with } a \in S,\ k \in K.$$

Exercise 1. A projection which is not orthogonal.

Here is one of these more general projections, one which is not orthogonal, in $\mathbb{R}^3$.

We are trying to write its matrix down, and figure out wether we can drop one of the requirements in the orthogonal projection definition above. Idempotence or symmetry? Can you already guess which one is more essential to the projection? which one we may drop?

Let us have

$$k_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad k_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and } e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We want P to be the projection along the space K spanned by k_1 and k_2, onto the first axis, that is the line spanned by e_1.

$$K = Vec(\{k_1, k_2\}), \quad S = Vec(\{e_1\})$$

First try and make a sketch in $\mathbb{R}^3$.

What is $Ker(P)$, the kernel (or Nullspace) of P?

What is $Col(P)$, the column space (or image) of P?

Recalling what lies in each column of P, just write P down.

Do we have $P^2 = P$, $P^T = P$?
Try to explain.

Exercise 2. A simple orthogonal projection.

Here is a very natural and simple to express orthogonal projection in $\mathbb{R}^3$.

Let P be the projection along the first axis, onto the plane containing the other two axes, i.e.

$$K = Vec(\{e_1\}), \quad S = Vec(\{e_2, e_3\})$$

Make a sketch.
Write down P.
Is it idempotent? Symmetric?
What is the rank of P?

Exercise 3. An orthogonal projection, seen as a column-row multiplication.

Here is another orthogonal projection in $\mathbb{R}^3$. It is rank two, and we are going to express it as the sum of two rank one projections.

Let

$$k_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad a_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \text{and } a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

and say P is the projection along the line spanned by k_1 onto the plan spanned by a_1 and a_2

$$K = Vec(\{k_1\}), \quad S = Vec(\{a_1, a_2\})$$

Recall the formula for the projection matrix,

$$\text{with} \quad A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}, \quad \text{we have} \quad P = A(A^T A)^{-1} A^T$$

trying to deduce it from the orthogonality constraints on the error

$$e = Ax - b \perp A \qquad \text{(that is, to each column of A)}$$

Now, first compute $(A^T A)^{-1}$. What happens here?
Can you see that what is left of P is the sum of two rank one projections

$$P = a_1 a_1^T + a_2 a_2^T$$

and is indeed a colum-row multiplication?