Homework 8

Subspaces.

A subspace S of a vector space V is a set of vectors belonging to V that

i) contains the zero vector

ii) is closed

When we require it to be closed, we obviously mean with respect to vector space operations, i.e. multiplication by a scalar and vector addition. Strictly speaking we do not need to state that it contains the zero vector, as we could deduce it from the closedness. But checking that a set of vectors contains the origin is the first thing to do if asked whether a given set S is a subspace, and we like to remind it at the beginning of the definition.

Exercise 1. Subspaces of $\mathbb{R}^3$

Describe all the possible subspaces of $\mathbb{R}^3$, maybe ordering them from the smallest to the biggest ones, thus starting with the *trivial subspace*, $S = \{0\}$, which only contains the null vector and is of size $\dim S = 0$.

Exercise 2. Creating new Subspaces with Set Operations.

A way to create new subspaces, if you already have some subspaces that you know of, is combining them somehow. If you answer yes to the following questions, sketch a proof, else find a counter example:

i) Is the *intersection* of two subspaces always a subspace?

ii) Is the *union* of two subspaces always a subspace?

Exercise 3. Boottom-Up and Top-Down.

Broadly speaking, there is two ways to build subspaces out of vectors.

- *Bottom-up* starting with a few vectors and keeping on adding linear combinations untill you have filled a whole subspace.
- *Top-down* that is starting with the whole space and ruling out sets of vectors using constraints, leaving a smaller space which is closed.

Here are four subspaces of $\mathbb{R}^3$, built with either one of these methods. Sketch them, and rebuild them using the other method. As usual, e_1, e_2 and e_3 are the componants of the canonical basis, that is the colums of the identity matrix I_3, and $Vec(\{u, v\})$ is the vector space spanned by the set of vectors containing u and v.

$$P_1 = Vec(\{e_1, e_2\}), \qquad L_2 = Vec(\{e_1 + e_2\}),$$

$$S_3 = \{x \in \mathbb{R}^3, x^T e_1 = 0\},$$

$$S_4 = \{x \in \mathbb{R}^3, Ax = 0\} \qquad \text{with} \quad A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$