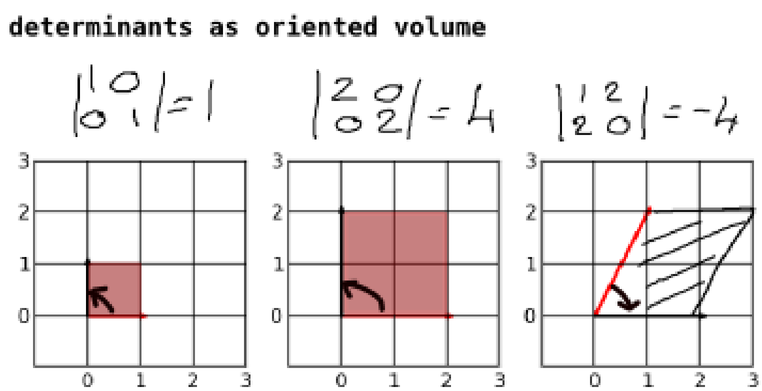
Orthogonal Matrices, Permutations

An orthogonal matrix is the matrix of an *isometric transform* of $\mathbb{R}^n$, that is one preserving sizes and angles. Isometries of $\mathbb{R}^n$ are rotations, reflections, and combinations of rotations and reflections. The orientation of some angles might be reversed, inverting the sign so, strictly speaking, it is the absolute value of the angles which is preserved. Because it preserves sizes, it also preserves volumes.

So an orthogonal matrix Q, is one for which $\|Qv\| = \|v\|$, and $\angle QvQw = \pm\angle vw$ for all vectors v, w.

Exercise 1. Is preserving Volume enough?

The *determinant* of a matrix is the oriented volume of the image of the unit cube. By oriented we mean it could be negative. Think of a unit sock, with volume one, which is pulled inside out thanks to a reflexion through some plane. Then we say that the volume of the image sock is negative one.



We will not get into much details about determinants here. You can just compute them using matlab or R, with `det(A)`, and then take the absolute value to get the "unoriented" volume, or else in the 2×2 case use the formula

$$\det(A) = ad - bc, \qquad \text{for} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Sketch the transform of the unit cube, a square in $\mathbb{R}^2$, for the following matrices and tell whether they preserve lengths, angles, angles orientation or volumes.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 1 \\ 0 & 1/2 \end{bmatrix}, \qquad E = \begin{bmatrix} 2 & 1 \\ 0 & -1/2 \end{bmatrix}$$

Is any volume preserving linear transform M, that is one with $|\det(M)| = 1$, also an isometry?

Exercise 2. A Definition of Orthogonality.

Here is a way to build the matrix Q of an isometry. In $\mathbb{R}^n$, take n independent unit vectors which are orthogonal to each other, that is an orthonormal basis of $\mathbb{R}^n$, put them in the columns of Q, and you have an orthogonal matrix.

Assuming you have just built a matrix Q this way, check that you have

$$Q^T Q = I = QQ^T$$

Now, in fact this equality is usually stated as the definition of an orthogonal matrix.

Taking the above formula as the definition of an orthogonal matrix, check that it is an isometry indeed. That is for any $v, w \in \mathbb{R}^n$, we have

$\|Qv\| = \|v\|$, and

$|\angle QvQw| = |\angle vw|$

(recalling that both length and angles come from the inner product, $\langle u, v\rangle = v^T u$).

Exercise 3. Permutations

A permutation matrix P is a square matrix with exactly one 1 on each of its rows and each of its colums, all other entries beeing 0. In other words, it is the identity matrix I with its colums shuffled.

Are permutations orthogonal, that is isometries? (three words suffice)

Given the matrix P of a permutation, choose the right matrix multiplication to explain the following.

What is the effect on the colums of A when the permutation multiplies on the right?

What is the effect on the rows of A when the inverse of P, that is P^T, multiplies on the left?

$$\widetilde{A} = AP$$
$$\widetilde{A} = P^T A$$

In $\mathbb{R}^3$, give the matrix of one of each of the following transforms (that is a numerical example) if it exists, or say why such a transform does not exists, when impossible to do so:

A permutation which is a rotation.

A permutation which is a reflexion.

A rotation which is not a permutation.

A permutation which is neither a reflexion, nor a rotation.