Homework 4

LU (Gauss-Jordan) and Inverses

Exercise 1. Gauss-Jordan works.

Explain why Gauss-Jordan works, and why it gives the inverse of a matrix, say $A \in \mathbb{R}^{3\times 3}$.

Write out the elimination steps in matrix form, assuming A is indeed invertible and no row exchanges are necessary, with for example

E_{ij} the matrix forward eliminating the a_{ij} entry below the diagonal in A,

D_i the diagonal matrix factoring out the pivot in row i (see exercise 3 below) and

F_{ij} the matrix doing backward elimination of the a_{ij} entry above the diagonal in A.

Try to convince yourself that each of these steps is produced by an invertible matrix, and hence the whole product, say E, is itself invertible (maybe no need to make full proofs here, explaining with words why it should work is enough...)

Start with the augmented matrix

$$\left[A | I_3\right]$$

(This is nothing new, it is explained in the last part of video lecture 3. Just go through the algebra again, maybe with a bit more details than in the video, where E is not decomposed in each of its steps).

Exercise 2. Inversion by Gauss-Jordan.

Which ones of the following matrices have an inverse? When it does exist, give the inverse matrix.

First recall why finding an $x \neq 0$ such that $Ax = 0$ means A is singular (non invertible).

So, looking at the columns of the following matrices, try to spot a dependent column, and hence an $x \neq 0$ s.t. $Ax = 0$. Else compute the inverse with Gauss-Jordan

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Exercise 3. LU Factorization for a symmetric Matrix.

Here is a symmetric matrix $A \in \mathbb{R}^{3\times 3}$, that is we have

$$A = A^T, \qquad \text{or} \qquad a_{ij} = a_{ji} \quad \text{(for all its entries)}.$$

Symmetric matrices are important. They have good eigenvalue properties and they show up a lot in applications.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 12 \end{bmatrix}.$$

Do the LU decomposition of A to get

$$A = LU$$

Now, it is a fact of life that you can factor out multiples appearing on the rows of any matrix, by multiplying it on the left with a diagonal matrix containing the factors. Convince yourself that it works, looking at the following example:

$$\begin{bmatrix} 2 & 4 & 6 \\ 7 & 7 & 7 \\ 15 & 10 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Now, try to decompose A further, extracting the pivots on the diagonal of U into their own diagonal matrix D, so that you will have

$$A = LD\widetilde{U}$$

And now look carefully at L and $\widetilde{U}$, what do you notice?

Is it an accident, or should it be so?

In case you have not seen it before, or forgot about it, the transpose of a product is the product of the transposes, but in reverse order:

$$(CBA)^T = A^T B^T C^T$$