Homework 3

Matrix Multiplication

Exercise 1. Four Ways to multiply BA.

Let B and A be the following matrices in $\mathbb{R}^{2\times 2}$.

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

i) Briefly recall the four ways to multiply two matrices.

ii) Compute the product BA in each of these four ways, expliciting the intermediate computations if possible.

iii) Recall the rules of matrix bloc decomposition for the product, that is where should the bloc sizes match for the multiplication to be valid. Now show that the four ways to do matrix multiplication can be seen as special cases of the block multiplication.

Exercise 2. Some 2×2 Matrices that commute.

Generally it is not true that $AB = BA$. Sometimes though, for a given pair of matrices A and B the equality holds, and we say that they *commute*.

i) It is easy to find two matrices in $\mathbb{R}^{2\times 2}$ that do not commute, especially if they are not invertible. For example

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Find two invertible matrices in $\mathbb{R}^{2\times 2}$, that is two matrices whose colums are independent, that do not commute. In the plane, sketch the geometric transform of each of your matrices and give some geometric insight as to why they do not commute.

ii) Some subsets of the square matrices are closed under multiplication. If we take two matrices from this subset and multiply them, we stay inside the subset. For example the set of *diagonal matrices* as well as the set of *upper triangular matrices* are closed. In $\mathbb{R}^{2\times 2}$ they have the following shape

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \qquad U = \begin{bmatrix} d_1 & a \\ 0 & d_2 \end{bmatrix}$$

Show that both the set of diagonal matrices and the set of upper triangular matrices in $\mathbb{R}^{2\times 2}$ are closed under multiplication. Is multiplication commutative inside these subsets?

Exercise 3. Powers of Matrices.

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \qquad \text{and } C = \begin{bmatrix} 1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$$

Compute $A^2, A^3, \ldots, A^n$, $B^2, B^3, \ldots, B^n$ and $C^2, C^3, \ldots, C^n$.