Homework 15

Eigenvalues

Exercise 1. True or false?

Let A and B be square $n \times n$ matrices with real entries, and $\lambda_1, \ldots, \lambda_p$ be the distinct eigenvalues of A (real or complex numbers). For each of the following assertions, if it is true prove it, else find a counter example. The last two questions might prove harder right now. If you want, you can just experiment on random matrices and guess whether the assertions are true or false.

i) $\lambda_1^2, \ldots, \lambda_p^2$ are also eigenvalues of A^2.

ii) $1/\lambda_1, \ldots, 1/\lambda_p$ are also eigenvalues of A^{-1} whenever A is invertible.

iii) If λ is an eigenvalue of A, it is also an eigenvalue of BA.

iv) If v is an eigenvector of A, it is also an eigenvector of BA.

v) If A is symmetric, that is $A = A^T$, then all the eigenvalues of A are real.

vi) A permutation matrix P can only have three eigenvalues, 0, 1 and -1.

Exercise 2. The eigenvalues of a 2×2 Matrix.

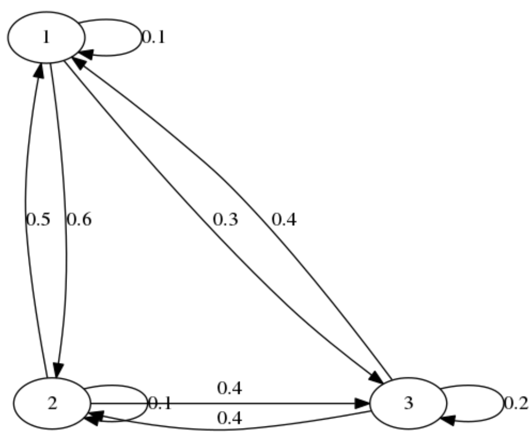
Let

$$A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$$

Compute the eigenvalues and the eigenvectors of A, A^2 and A^{-1} for $\alpha = 0$, $\alpha = 1$, and $\alpha = 2$. Do as few computations as you can, using the results of Exercise 1. Check your results with Matlab/Julia/Mathematica/etc...

Exercise 3. A Markov Chain.

A frog is living in a pond with three lilies named 1, 2 and 3. Each lily has its own coin attached to it. Because the coins are very old and bent, they are not only unfair but can land on their edge. Each morning the frog flips the coin attached to the lily it has spent the last 24 hours on, say lily j, and based on the outcome decides where to move next. If the coin lands heads, it moves to lily $j+1$ (or 1 if is on 3), if the coin lands tails it moves to lily $j-1$ (or 3 if it is on 1) and if the coin lands on the edge it stays one more day on the same lily. This can be represented as a weighted directed graph with three nodes and nine edges. Each edge's weight out of lily j is the probability that the frog will follow this edge the next morning, if it is on lily j currently:



It can also be represented as a matrix M where entry m_{ij} is the probability to transition from node j to node i each morning. For example the first column wil contain the weights of the edges leaving node 1 (including the loop with weight 0.1 staying on 1).

$$M = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.1 & 0.4 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

This is a Markov matrix, the entries in each of its colums add to one. Let us note $p^{(k)}$ the probability distribution of the location of the frog after k days. It is a random vector in $\mathbb{R}^3$ with all its entries between 0 and 1 and $p_i^{(k)}$ is the probability to be on lily i on day k. For example $p^{(k)} = (1/3, 1/3, 1/3)^T$ means we do not know anything about the location of the frog on day k, because it has a uniform probability of 1/3 to be located at any lily. Let us assume for the moment that we know the location on day 0 for sure, and so $p^{(0)}$ is the vector $(1, 0, 0)^T$ (meaning the frog is on lily 1). For the following computations, use Matlab/Julia or some other computing environment of your choice.

i) What is $p^{(1)} = Mp^{(0)}$? What is $p^{(2)} = Mp^{(1)} = M^2 p^{(0)}$?

ii) What are $p^{(19)} = M^{19} p^{(0)}$ and $p^{(20)} = M^{20} p^{(0)}$?

iii) Now change $p^{(0)}$ to $(0, 0, 1)^T$ and answer the two preceding questions again? What happens? Look at the three colums of M^{19}, what do you notice? Why should it be so.

iv) compute the eigenvalues of M.

Write a Matlab/Julia script to generate random Markov matrices of size n, that is matrices for which $0 \leq m_{ij} \leq 1$ holds, and where the entries of each column add to one. For a few values of n not too big, say $n = 3, 4$ and 5, generate a dozen different Markov matrices and have their eigenvalues computed. What do you notice? Could it be a coincidence or must it be so? Try to prove your conjecture.

A note on notation. In most of the probability books, Markov matrices are written as row matrices, not as column matrices. So the first row contains the transition probabilities from state 1 to each of the n states, and all the linear algebra gets reversed. Entry m_{ij} is the probability to transition from node i to node j. Probability distribution vectors are row vectors and they multiply matrices from left to right

$$p_{(1)}^T = p_{(0)}^T M$$